J Eng Math (2010) 66:141–152
DOI 10.1007/s10665-009-9326-7

Restricted Lie point symmetries and reductions for ideal magnetohydrodynamics equilibria

Oleg Bogoyavlenskij

Received: 31 December 2008 / Accepted: 28 August 2009 / Published online: 29 September 2009

Abstract Restricted Lie point symmetries are derived for the axially symmetric steady solutions to the ideal magnetohydrodynamics equations. The symmetries transform vectors of magnetic field B and plasma velocity V linearly with coefficients depending on a function $u(z, r)$. A reduction of the eight MHD equilibrium equations to a single second-order partial differential equation for the function $u(z, r)$ is obtained. Analogous Lie point symmetries and reduction are derived for the translationally invariant MHD equilibria. Applications of the symmetry transforms are indicated.

Keywords Axially symmetric MHD equilibria · Grad–Shafranov equation · Minkowski metric · Poisson brackets

1 Introduction

The study of the ideal magnetohydrodynamics (MHD) equilibrium equations has been going on during the last four decades since their first applications to the problem of controlled thermonuclear fusion [1–4] and to problems in astrophysics [5,6]. In this paper, we study the intrinsic symmetries of these equations which were introduced for the first time in [7–9]. Intrinsic symmetries (or Bäcklund transforms) exist for all soliton equations [10–12], such as the Korteweg–de Vries equation, the Kadomtzev–Petviashvili equation, the Sine–Gordon equation, etc. For the soliton equations, the Bäcklund transforms are given implicitly and do not have explicit algebraic form.

We present the eight MHD equilibrium equations for the axially symmetric and translationally symmetric solutions in an algebraic form in terms of Poisson brackets of different functions. Using this algebraic form, we derive the restricted Lie point symmetries that are applicable to the physically meaningful MHD equilibria with bounded total energy in any layer $z_1 < z < z_2$. Unlike the Bäcklund transforms for the soliton equations, the new symmetries are given explicitly. The symmetries transform the vectors of plasma velocity V and magnetic field B by linear formulae with variable coefficients and preserve the axial and translational invariance of the MHD equilibria.

The symmetries differ from the general symmetry transforms introduced in [7–9]. The latter are applicable to any MHD equilibria and can break the geometrical symmetry of solutions. Using the known properties of the Poisson brackets, we show that eight MHD equations are reduced to a single second-order partial differential equation for

O. Bogoyavlenskij (✉)
Queen's University, Kingston, ON K7L 3N6, Canada
e-mail: Bogoyavl@mast.queensu.ca

a function $u(z, r)$. For the axially symmetric case, the equation contains three arbitrary functions and generalizes the Grad–Shafranov equation [3,4] for purely magnetic plasma equilibria.

2 Restricted Lie point symmetries for axially symmetric MHD equilibria

2.1 Lie point symmetries in exact form

Equations of ideal magnetohydrodynamics have the form [13, Chap. 1]

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu\rho}\mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad}\left(\frac{p}{\rho} + \frac{\mathbf{V}^2}{2}\right), \tag{2.1}$$

$$\operatorname{div} \mathbf{V} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}), \quad \operatorname{div} \mathbf{B} = 0, \tag{2.2}$$

where $\mathbf{B}$ is the magnetic vector field, $\mathbf{V}$ is plasma velocity vector field and p denotes plasma pressure; we assume that magnetic permeability μ and plasma density ρ are constant. First we study axially symmetric solutions which depend on three variables t, z, r in the cylindrical coordinates t, z, r, φ. Any axially symmetric vector field $\mathbf{U}$ has the form

$$\mathbf{U} = \frac{X(t, z, r)}{r}\mathbf{e}_z + \frac{Y(t, z, r)}{r}\mathbf{e}_r + \frac{f(t, z, r)}{r}\mathbf{e}_\varphi$$

where $\mathbf{e}_z$, $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are unit orts in the directions of variables z, r and φ, respectively. The incompressibility equation $\operatorname{div} \mathbf{U} = 0$ has the form $\partial X/\partial z + \partial Y/\partial r = 0$. Hence the vector fields $\mathbf{V}$ and $\mathbf{B}$ satisfying equations $\operatorname{div} \mathbf{V} = 0$ and $\operatorname{div} \mathbf{B} = 0$ have the form

$$\mathbf{V} = -\frac{v_r}{r}\mathbf{e}_z + \frac{v_z}{r}\mathbf{e}_r + \frac{f}{r}\mathbf{e}_\varphi, \quad B = -\frac{h_r}{r}\mathbf{e}_z + \frac{h_z}{r}\mathbf{e}_r + \frac{g}{r}\mathbf{e}_\varphi, \tag{2.3}$$

where $v(t, z, r),\ f(t, z, r),\ h(t, z, r)$ and $g(t, z, r)$ are some smooth functions. The vector field $\mathbf{V}$ has the form

$$\mathbf{V} = \left(\frac{x}{r^2}v_z - \frac{y}{r^2}f\right)\mathbf{e}_x + \left(\frac{y}{r^2}v_z + \frac{x}{r^2}f\right)\mathbf{e}_y - \frac{v_r}{r}\mathbf{e}_z$$

in the Cartesian coordinates x, y, z with unit orts $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. Hence we get $\mathbf{V}^2 = ((v_z)^2 + (v_r)^2 + f^2)/r^2$.

The vorticity vector fields $\operatorname{curl} \mathbf{V}$ and $\operatorname{curl} \mathbf{B}$ are

$$\operatorname{curl} \mathbf{V} = \frac{f_r}{r}\mathbf{e}_z - \frac{f_z}{r}\mathbf{e}_r + r\Phi\mathbf{e}_\varphi, \quad \operatorname{curl} \mathbf{B} = \frac{g_r}{r}\mathbf{e}_z - \frac{g_z}{r}\mathbf{e}_r + r\Psi\mathbf{e}_\varphi,$$

where

$$\Phi = \frac{1}{r^2}\frac{\partial^2 v}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial r}\right), \quad \Psi = \frac{1}{r^2}\frac{\partial^2 h}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial h}{\partial r}\right). \tag{2.4}$$

We will use the notion of the Poisson bracket $\{F, G\} = F_z G_r - F_r G_z$ of two functions $F(t, z, r)$ and $G(t, z, r)$ and the main identities

$$\{F, G\} = -\{G, F\}, \quad \{F, f(u)\} = f'(u)\{F, u\},$$
$$\{FG, H\} = F\{G, H\} + G\{F, H\}.$$

The $z-$, $r-$, and φ-components of Eq. 2.1 have the form

$$\frac{1}{r}v_{tr} + \frac{1}{2r^2}D_z + \Phi v_z - \frac{1}{\mu\rho}\Psi h_z = \mathcal{P}_z, \tag{2.5}$$

$$-\frac{1}{r}v_{tz} + \frac{1}{2r^2}D_r + \Phi v_r - \frac{1}{\mu\rho}\Psi h_r = \mathcal{P}_r, \tag{2.6}$$

$$f_t = \frac{1}{r}\left(\{f, v\} - \frac{1}{\mu\rho}\{g, h\}\right), \tag{2.7}$$

where $D = f^2 - g^2/\mu\rho$ and $\mathcal{P} = p/\rho + \mathbf{V}^2/2$. The compatibility condition for (2.5) and (2.6) is

$$\Phi_t = \frac{1}{r}\left(\{\Phi, v\} - \frac{1}{\mu\rho}\{\Psi, h\} + \left\{\frac{1}{2r^2}, D\right\}\right). \tag{2.8}$$

The $r-$, $z-$, and φ-components of the second equation of (2.2) have the form

$$\frac{\partial}{\partial z}\left(h_t - \frac{1}{r}\{h, v\}\right) = 0, \quad \frac{\partial}{\partial r}\left(h_t - \frac{1}{r}\{h, v\}\right) = 0, \tag{2.9}$$

$$g_t = r(\{r^{-2}g, v\} - \{r^{-2}f, h\}). \tag{2.10}$$

Equations 2.9 mean that $h_t - \{h, v\}/r = c(t)$. Solutions to (2.1)–(2.2) are unchanged after the substitution $h \longrightarrow h + c_1(t)$ with $\mathrm{d}c_1(t)/\mathrm{d}t = c(t)$. Hence the equation set (2.9) is reduced to

$$h_t = \frac{1}{r}\{h, v\}. \tag{2.11}$$

The four equations (2.7), (2.8), (2.10) and (2.11) define completely solutions that depend on three variables t, z, r.

Remark 1 For the time-independent solutions, Eq. 2.9 yields

$$\frac{1}{r}\{h, v\} = K = \text{const.} \tag{2.12}$$

Using Schwarz' inequality and formulae (2.3), we find

$$|K| = \frac{|h_z v_r - h_r v_z|}{r} \le \frac{\sqrt{\mu\rho}}{2r}\left(v_r^2 + v_z^2 + \frac{h_r^2 + h_z^2}{\mu\rho}\right) \le \frac{r\sqrt{\mu\rho}}{2}\left(V^2 + \frac{B^2}{\mu\rho}\right). \tag{2.13}$$

Let us consider the physically meaningful MHD equilibria for which the total kinetic and magnetic energy of plasma is finite in every axially symmetric layer $z_1 < z < z_2$:

$$\frac{1}{2}\int_{z_1}^{z_2} \mathrm{d}z \int \int_{-\infty}^{\infty} (\rho\mathbf{V}^2 + \mathbf{B}^2/\mu)\mathrm{d}x\mathrm{d}y = \int_{z_1}^{z_2} \mathrm{d}z \int_0^{\infty} \pi r(\rho\mathbf{V}^2 + \mathbf{B}^2/\mu)\mathrm{d}r < \text{const.} \tag{2.14}$$

Hence we get that at least for some sequence of $r \longrightarrow \infty$

$$r(\mathbf{V}^2 + \mathbf{B}^2/\mu\rho) \longrightarrow 0.$$

Therefore the inequalities (2.13) imply that for physically meaningful MHD equilibria necessarily $K = 0$. Hence we get from (2.12) the equation

$$\{h, v\} = 0 \tag{2.15}$$

for the considered MHD solutions.

Remark 2 The equation $\{h, v\} = h_z v_r - h_r v_z = 0$ means that the Jacobian of the mapping

$$(z, r) \longrightarrow (h(z, r), v(z, r))$$

is identically equal to zero. Hence the image of this mapping is a curve Γ in the plane (h, v) and the stream function $v(z, r)$ and the flux function $h(z, r)$ are functionally dependent. Therefore $v = v(u)$, $h = h(u)$, where $u = u(z, r)$ is an unknown function. Let the curve Γ satisfies an equation such as

$$S(h, v) = 0. \tag{2.16}$$

Thus, the space of all solutions to the steady equations (2.1)–(2.2) is mapped into a much smaller space of plane curves (2.16). This means that the curve Γ defined by (2.16) is a geometric invariant of any magnetohydrodynamics equilibrium configuration. Any two distinct curves (2.16) correspond to different solutions to (2.1)–(2.2).

Theorem 1 *The steady equations of magnetohydrodynamics have the following Lie point symmetry applicable to the axially symmetric equilibria satisfying the physical condition (2.14). Let vector fields* $\mathbf{V}(z,r)$ *and* $\mathbf{B}(z,r)$ *(2.3) and pressure* $p(z,r)$ *satisfy the steady equations (2.1)–(2.2). Let* $v_1(u)$ *and* $h_1(u)$ *be arbitrary smooth functions satisfying the equation*

$$v_1'^2(u) - \frac{1}{\mu\rho} h_1'^2(u) = C\left(v'^2(u) - \frac{1}{\mu\rho} h'^2(u)\right), \tag{2.17}$$

where $C \neq 0$ *is an arbitrary constant. The symmetry transforms solution* $\mathbf{V}(z,r)$, $\mathbf{B}(z,r)$, $p(z,r)$ *into the new solution*

$$\begin{aligned} \mathbf{V}_1 &= \frac{v_1'}{v'}\mathbf{V} + \frac{v_1'h' - v'h_1'}{m\mu\rho r v'}(h'f - v'g)\mathbf{e}_\varphi, \\ \mathbf{B}_1 &= \frac{h_1'}{h'}\mathbf{B} - \frac{v_1'h' - v'h_1'}{mrh'}(h'f - v'g)\mathbf{e}_\varphi, \\ p_1 &= Cp + \frac{\rho}{2}(C\mathbf{V}^2 - \mathbf{V}_1^2), \end{aligned} \tag{2.18}$$

where $m = v'^2(u) - h'^2(u)/\mu\rho$.

Proof Indeed, Eqs. (2.7)–(2.10), (2.15) for the steady case take the form

$$\{h, v\} = 0, \quad \{\Phi, v\} - \frac{1}{\mu\rho}\{\Psi, h\} + \left\{\frac{1}{2r^2}, f^2 - \frac{1}{\mu\rho} g^2\right\} = 0, \tag{2.19}$$

$$\{f, v\} - \frac{1}{\mu\rho}\{g, h\} = 0, \quad \{r^{-2}g, v\} - \{r^{-2}f, h\} = 0. \tag{2.20}$$

Suppose that (2.16) is resolved in a parametric form $h = h(u)$, $v = v(u)$ where $u = u(z,r)$. Then the second equation of (2.19) takes the form

$$\begin{aligned} &\left(v'^2 - \frac{1}{\mu\rho}h'^2\right)\left\{\frac{1}{r^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right), u\right\} + \left(v'v'' - \frac{1}{\mu\rho}h'h''\right)\left\{\frac{1}{r^2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial r}\right)^2, u\right\} \\ &\quad + \left\{\frac{1}{2r^2}, f^2 - \frac{1}{\mu\rho}g^2\right\} = 0. \end{aligned} \tag{2.21}$$

Let us consider (2.19)–(2.20) for another four functions $v_1(u)$, $h_1(u)$, $f_1(z,r)$, $g_1(z,r)$ instead of the functions $v(u)$, $h(u)$, $f(z,r)$ and $g(z,r)$. The second equation of (2.19) for them has the form

$$\begin{aligned} &\left(v_1'^2 - \frac{1}{\mu\rho}h_1'^2\right)\left\{\frac{1}{r^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right), u\right\} + \left(v_1'v_1'' - \frac{1}{\mu\rho}h_1'h_1''\right)\left\{\frac{1}{r^2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial r}\right)^2, u\right\} \\ &\quad + \left\{\frac{1}{2r^2}, f_1^2 - \frac{1}{\mu\rho}g_1^2\right\} = 0. \end{aligned}$$

This equation is equivalent to (2.21) if the functions $v_1(u)$, $h_1(u)$ satisfy (2.17) and the functions $f_1(z,r)$, $g_1(z,r)$ satisfy

$$f_1^2 - \frac{1}{\mu\rho}g_1^2 = C\left(f^2 - \frac{1}{\mu\rho}g^2\right). \tag{2.22}$$

Equations (2.20) for the functions v_1, h_1, f_1, g_1 take the form

$$\left\{v_1'f_1 - \frac{1}{\mu\rho}h_1'g_1, u\right\} = 0, \quad \left\{r^{-2}(v_1'g_1 - h_1'f_1), u\right\} = 0. \tag{2.23}$$

Equations (2.23) are equivalent to the original equations (2.20) if

$$v_1'f_1 - \frac{1}{\mu\rho}h_1'g_1 = C\left(v'f - \frac{1}{\mu\rho}h'g\right), \quad v_1'g_1 - h_1'f_1 = C\left(v'g - h'f\right). \tag{2.24}$$

Solving (2.24), we obtain

$$\begin{aligned} f_1 &= \frac{C}{m_1}\left(\left(v'v_1' - \frac{1}{\mu\rho}h'h_1'\right) f + \frac{1}{\mu\rho}\left(v'h_1' - h'v_1'\right) g\right),\\ g_1 &= \frac{C}{m_1}\left(\left(v'h_1' - h'v_1'\right) f + \left(v'v_1' - \frac{1}{\mu\rho}h'h_1'\right) g\right), \end{aligned} \tag{2.25}$$

where $m_1 = v_1'^2 - h_1'^2/\mu\rho$. A direct verification proves that the functions f_1, g_1 (2.25) do satisfy (2.22) if (2.17) holds.

For any given steady solution $\mathbf{V}(z, r)$, $\mathbf{B}(z, r)$, the obtained results prove the following. If one changes in formulae (2.3) the functions $v(u)$, $h(u)$, $f(z, r)$, $g(z, r)$ to new functions $v_1(u)$, $h_1(u)$, $f_1(z, r)$, $g_1(z, r)$ satisfying (2.17) and (2.25), then the new vector fields $\mathbf{V}_1(z, r)$, $\mathbf{B}_1(z, r)$ form a solution to (2.1), (2.2). This transformation has the exact form (2.18) and defines a symmetry transform for the steady equations (2.1)–(2.2), (2.15).

To derive formula (2.18) for the pressure $p_1(z, r)$, we consider the steady equations (2.5), (2.6) for the new functions $v_1(u)$, $h_1(u)$, $f_1(z, r)$, $g_1(z, r)$. Using formulae (2.4), (2.17) and (2.22), we find $\mathcal{P}_1 = C\mathcal{P}$:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = C\left(\frac{p}{\rho} + \frac{V^2}{2}\right).$$

Hence formula (2.18) for the pressure $p_1(z, r)$ follows. □

Remark 3 The symmetry transforms (2.18) have efficient applications. Indeed, to obtain large families of exact axially symmetric MHD equilibria it is sufficient to apply the symmetries (2.18) to the exact plasma equilibria derived in [14,15].

2.2 Algebraic properties of symmetries

Using (2.3) and (2.15), which means $v = v(u)$, $h = h(u)$, we find

$$h'\mathbf{V} - v'\mathbf{B} = \frac{1}{r}(h'f - v'g)\mathbf{e}_\varphi. \tag{2.26}$$

Excluding vector $\mathbf{e}_\varphi$ from (2.26) and inserting it into (2.18), we obtain the following formulae for the symmetry transform

$$\begin{aligned} \mathbf{V}_1 &= \frac{1}{m}\left(\left(v'v_1' - \frac{1}{\mu\rho}h'h_1'\right)\mathbf{V} + \frac{1}{\mu\rho}\left(v'h_1' - h'v_1'\right)\mathbf{B}\right),\\ \mathbf{B}_1 &= \frac{1}{m}\left(\left(v'h_1' - h'v_1'\right)\mathbf{V} + \left(v'v_1' - \frac{1}{\mu\rho}h'h_1'\right)\mathbf{B}\right). \end{aligned} \tag{2.27}$$

It is evident that transform (2.27) is linear with respect to vectors $\mathbf{V}$ and $\mathbf{B}$. Using the equation $m_1 = Cm$ (2.17) where $m_1 = v_1'^2 - h_1'^2/\mu\rho$, we see that linear transform (2.27) coincides with that of equations (2.25).

A direct calculation shows that a combination of transform (2.27) and the transform

$$\begin{aligned} \mathbf{V}_2 &= \frac{1}{m_1}\left(\left(v_1'v_2' - \frac{1}{\mu\rho}h_1'h_2'\right)\mathbf{V}_1 + \frac{1}{\mu\rho}\left(v_1'h_2' - h_1'v_2'\right)\mathbf{B}_1\right),\\ \mathbf{B}_2 &= \frac{1}{m_1}\left(\left(v_1'h_2' - h_1'v_2'\right)\mathbf{V}_1 + \left(v_1'v_2' - \frac{1}{\mu\rho}h_1'h_2'\right)\mathbf{B}_1\right) \end{aligned} \tag{2.28}$$

gives

$$\begin{aligned} \mathbf{V}_2 &= \tfrac{1}{m}\left(\left(v'v_2' - \tfrac{1}{\mu\rho}h'h_2'\right)\mathbf{V} + \tfrac{1}{\mu\rho}\left(v'h_2' - h'v_2'\right)\mathbf{B}\right),\\ \mathbf{B}_2 &= \tfrac{1}{m}\left(\left(v'h_2' - h'v_2'\right)\mathbf{V} + \left(v'v_2' - \tfrac{1}{\mu\rho}h'h_2'\right)\mathbf{B}\right). \end{aligned} \tag{2.29}$$

It is evident that the resulting transform (2.29) has the same form as (2.27).

Remark 4 The transforms (2.27) are parameterized by the mappings of curves the Γ (2.16) in the Minkowski plane R^2 with metric $ds^2 = dv^2 - dh^2/\mu\rho$:

$(v(u), h(u)) \longrightarrow (v_1(u), h_1(u))$

satisfying (2.17). The transforms do not form a Lie group because their product is not defined for any two mappings but only if they have the form $(v(u), h(u)) \longrightarrow (v_1(u), h_1(u))$ and $(v_1(u), h_1(u)) \longrightarrow (v_2(u), h_2(u))$. The unit transform is represented by the identity map $(v(u), h(u)) \longrightarrow (v(u), h(u))$; the inverse transform is represented by the mapping $(v_1(u), h_1(u)) \longrightarrow (v(u), h(u))$. The associativity of multiplication (when it is defined) follows from the composition formula (2.29).

Formulae (2.27) and $m_1 = Cm$ (2.16) yield the following equation

$$\mathbf{V}_1^2 - \frac{1}{\mu\rho}\mathbf{B}_1^2 = C\left(\mathbf{V}^2 - \frac{1}{\mu\rho}\mathbf{B}^2\right). \tag{2.30}$$

Remark 5 The linear transformation (2.27) has the form

$$\mathbf{V}_1 = a(u)\mathbf{V} + \frac{b(u)}{\sqrt{\mu\rho}}\mathbf{B}, \quad \mathbf{B}_1 = b(u)\sqrt{\mu\rho}\,\mathbf{V} + a(u)\mathbf{B}, \tag{2.31}$$

$$a(u) = \frac{v'v_1' - h'h_1'}{v'^2 - h'^2/\mu\rho}, \quad b(u) = \frac{v'h_1' - h'v_1'}{\sqrt{\mu\rho}(v'^2 - h'^2/\mu\rho)}.$$

In view of Eq. 2.17, the determinant of the transform (2.31) is constant:

$$a^2(u) - b^2(u) = C = \text{const}. \tag{2.32}$$

Formulae (2.31) and (2.32) look the same as for the general symmetry transforms introduced in [8,9]. However, there is an important distinction: the symmetry transforms of [8,9] have coefficients $a(x)$ and $b(x)$ that are constant on magnetic surfaces but depend upon the transversal variable. Hence they do not preserve the axial symmetry of solutions since magnetic surfaces (tangent to the commuting vector fields $\mathbf{V}$ and $\mathbf{B}$) for the axially symmetric MHD equilibria are not axially symmetric in general. The symmetry transforms [8,9] are applicable to all MHD equilibria without any restrictions. For the transforms (2.27)–(2.31), the coefficients $a(u)$ and $b(u)$ depend on a function $u(z, r)$ and hence they are not constant on magnetic surfaces. The transforms (2.27)–(2.31) do preserve the axial symmetry of MHD equilibria and are applicable only under the restriction (2.14), (2.15).

2.3 Connections with Minkowski geometry

Let the functions $v_1(u)$ and $h_1(u)$ satisfy an equation of the kind

$$S_1(h_1, v_1) = 0. \tag{2.33}$$

It is evident that (2.17) for $C = 1$ means that two curves Γ (2.16) and Γ_1 (2.33) are isometric with respect to the Minkowski metric

$$ds^2 = dv^2 - \frac{1}{\mu\rho}dh^2$$

on the plane v, h.

For the Minkowski metric, the classes of non-isometric curves are much richer then those for the Euclidean metric where there is only one invariant, namely the length of the curve. Let $p_1, p_2, \ldots, p_n, \ldots$ be successive points where a curve Γ is tangent to the light cone and $\ell_1, \ell_2, \ldots, \ell_n, \ldots$ be the Minkowski-lengths of the curve Γ segments between those points:

$$\ell_i = \varepsilon_i \int_{p_{i-1}}^{p_i} \left| dv^2 - \frac{1}{\mu\rho}dh^2 \right|^{1/2}.$$

Here $\varepsilon_i = \text{sign}(ds^2)|_\Gamma$ on the segment (p_{i-1}, p_i). Two curves Γ and Γ_1 are Minkowski-isometric if their two respective sequences $\ell_1, \ell_2, \ldots, \ell_n, \ldots$ coincide.

3 Reduction of the steady MHD equations

Suppose that the equation set (2.15), (2.16) is resolved in a parametric form $h = h(u)$, $v = v(u)$. Then the equations of (2.20) take the form

$$\left\{v'(u)f - \frac{1}{\mu\rho}h'(u)g, u\right\} = 0, \quad \{r^{-2}(h'(u)f - v'(u)g), u\} = 0.$$

These equations imply

$$v'(u)f - \frac{1}{\mu\rho}h'(u)g = k(u), \quad h'(u)f - v'(u)g = -r^2\ell(u), \tag{3.1}$$

where $k(u)$ and $\ell(u)$ are some functions. Solving Eq. 3.1, we obtain

$$f = \frac{v'(u)k(u) + \frac{r^2}{\mu\rho}h'(u)\ell(u)}{m(u)}, \quad g = \frac{r^2 v'(u)\ell(u) + h'(u)k(u)}{m(u)}, \tag{3.2}$$

where $m(u) = (v'(u))^2 - (h'(u))^2/\mu\rho$.

The formulae (3.2) imply that function $D = f^2 - g^2/\mu\rho$ has the form

$$D = \lambda(u) - r^4\psi(u), \quad \lambda(u) = \frac{k^2(u)}{m(u)}, \quad \psi(u) = \frac{\ell^2(u)}{\mu\rho m(u)}. \tag{3.3}$$

Using these formulae, we transform the second equation of (2.19) to the form $\{Z, u\} = 0$, where

$$Z = v'(u)\Phi - \frac{h'(u)}{\mu\rho}\Psi + \frac{\lambda'(u)}{2r^2} + \frac{r^2\psi'(u)}{2}. \tag{3.4}$$

Hence we conclude as above that $Z = \eta(u)$, where the function $\eta(u)$ can be multivalued in general.

Applying formulae (2.4), we obtain the explicit form of the equation $Z = \eta(u)$:

$$\left(v'^2 - \frac{1}{\mu\rho}h'^2\right)\left(\frac{1}{r^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right)\right) + \left(v'v'' - \frac{1}{\mu\rho}h'h''\right)\left(\frac{1}{r^2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial r}\right)^2\right)$$
$$= \eta(u) - \frac{1}{2r^2}\lambda'(u) - \frac{r^2}{2}\psi'(u). \tag{3.5}$$

On any segment of the curve Γ (2.16) between two adjacent points u_j and u_{j+1} where $m(u_\ell) = 0$, we introduce a new parametrization satisfying the condition

$$(v'(u))^2 - \frac{1}{\mu\rho}(h'(u))^2 = \beta, \tag{3.6}$$

where $\beta = \text{const}$. The condition (3.6) reduces (3.5) to the form

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\eta(u)}{\beta} - \frac{1}{2\beta r^2}\lambda'(u) - \frac{r^2}{2\beta}\psi'(u). \tag{3.7}$$

Equation (3.7) generalizes the Grad–Shafranov equation [3,4] for purely magnetic plasma equilibria with $\mathbf{V} = 0$. For the Grad–Shafranov equation we have $f = 0$, $v(u) = \ell(u) = \psi(u) = 0$. In the coordinates $z, x = r^2/4$, Eq. 3.7 takes the form

$$u_{zz} + xu_{xx} = a(u) + xb(u) + x^2c(u), \tag{3.8}$$

where

$$a(u) = -\frac{(k^2(u))'}{2\beta^2}, \quad b(u) = \frac{4\eta(u)}{\beta}, \quad c(u) = -\frac{8(\ell^2(u))'}{\mu\rho\beta^2}.$$

Any solution to (3.8) defines a solution to the magnetohydrodynamics equations (2.1)–(2.2) by the formulae

$$\mathbf{V} = -\frac{v'u_r}{r}\mathbf{e}_z + \frac{v'u_z}{r}\mathbf{e}_r + \frac{f}{r}\mathbf{e}_\varphi, \tag{3.9}$$

$$\mathbf{B} = -\frac{h'u_r}{r}\mathbf{e}_z + \frac{h'u_z}{r}\mathbf{e}_r + \frac{g}{r}\mathbf{e}_\varphi, \tag{3.10}$$

where functions $v(u)$ and $h(u)$ satisfy (3.6) and

$$f = \frac{1}{\beta}\left(v'k + \frac{r^2}{\mu\rho}h'\ell\right), \quad g = \frac{1}{\beta}\left(r^2 v'\ell + h'k\right).$$

To derive a formula for the pressure $p(z, r)$, we consider the steady equations (2.5) and (2.6). Using the formulae (3.3) $D = \lambda(u) - r^4\psi(u)$ and (3.4) $Z = \eta(u)$, we transform (2.5) and (2.6) to the form

$$\eta(u)u_z - (r^2\psi(u))_z = \mathcal{P}_z, \quad \eta(u)u_r - (r^2\psi(u))_r = \mathcal{P}_r,$$

which is equivalent to $\mathcal{P} = F(u) - r^2\psi(u)$ where $\mathcal{P} = p/\rho + V^2/2$ and $F'(u) = \eta(u) = \beta b(u)/4$. Hence we obtain the following formula for the pressure p:

$$p = \rho F(u) - \frac{r^2\ell^2}{\mu\beta} - \frac{\rho}{2r^2}(f^2 + v'^2((u_z)^2 + (u_r)^2)). \tag{3.11}$$

A direct substitution of formulae (3.9)–(3.11) in the steady magnetohydrodynamics equations (2.1)–(2.2) proves that these eight equations for the axially symmetric solutions are reduced to the single equation (3.8) provided that condition (3.6) holds.

4 Restricted Lie point symmetries and reduction for translationally invariant MHD equilibria

4.1 Explicit formulae for the Lie point symmetries

Let us consider the z-independent (or translationally invariant) solutions to equations of ideal magnetohydrodynamics in the Cartesian coordinates t, x, y, z. First we study solutions which depend on three variables t, x, y. The incompressibility equations $\operatorname{div}\mathbf{V} = 0$ and $\operatorname{div}\mathbf{B} = 0$ imply that the vector fields $\mathbf{V}$ and $\mathbf{B}$ have the form

$$\mathbf{V} = -v_y\mathbf{e}_x + v_x\mathbf{e}_y + f\mathbf{e}_z, \quad \mathbf{B} = -h_y\mathbf{e}_x + h_x\mathbf{e}_y + g\mathbf{e}_z, \tag{4.1}$$

where $v(t, x, y)$, $f(t, x, y)$, $h(t, x, y)$ and $g(t, x, y)$ are some smooth functions. The corresponding vorticity vector fields are

$$\operatorname{curl}\mathbf{V} = f_y\mathbf{e}_x - f_x\mathbf{e}_y + \Phi\mathbf{e}_z, \quad \operatorname{curl}\mathbf{B} = g_y\mathbf{e}_x - g_x\mathbf{e}_y + \Psi\mathbf{e}_z,$$

where

$$\Phi = v_{xx} + v_{yy}, \quad \Psi = h_{xx} + h_{yy}. \tag{4.2}$$

We will use the notion of the Poisson bracket $\{F, G\} = F_xG_y - F_yG_x$ of two functions $F(t, x, y)$ and $G(t, x, y)$.

The x-, y-, and z-components of (2.1) have the form

$$v_{ty} + \frac{1}{2}D_x + \Phi v_x - \frac{1}{\mu\rho}\Psi h_x = \mathcal{P}_x, \tag{4.3}$$

$$-v_{tx} + \frac{1}{2}D_y + \Phi v_y - \frac{1}{\mu\rho}\Psi h_y = \mathcal{P}_y, \tag{4.4}$$

$$f_t = \{f, v\} - \frac{1}{\mu\rho}\{g, h\}, \tag{4.5}$$

where $D = f^2 - g^2/\mu\rho$ and $\mathcal{P} = p/\rho + \mathbf{V}^2/2$. The compatibility condition for (4.3) and (4.4) is

$$\Phi_t = \{\Phi, v\} - \frac{1}{\mu\rho}\{\Psi, h\}. \tag{4.6}$$

The x-, y-, and z-components of the second equation of (2.2) have the form

$$\frac{\partial}{\partial y}(h_t - \{h, v\}) = 0, \quad \frac{\partial}{\partial x}(h_t - \{h, v\}) = 0, \tag{4.7}$$

$$g_t = \{g, v\} - \{f, h\}. \tag{4.8}$$

The equations of (4.7) mean that $h_t - \{h, v\} = c(t)$. The solutions to (2.1)–(2.2) are unchanged after the substitution $h \longrightarrow h + c(t)$ with $\mathrm{d}c_1(t)/\mathrm{d}t = c(t)$. Hence (4.7) is reduced to

$$h_t = \{h, v\}. \tag{4.9}$$

Equations (4.6), (4.9) form a closed subsystem that has the form

$$\Delta v_t = \{\Delta v, v\} - \frac{1}{\mu\rho}\{\Delta h, h\}, \quad h_t = \{h, v\},$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The four equations (4.5), (4.6),(4.8) and (4.9) define completely the solutions which depend on the three variables t, x, y.

Remark 6 For time-independent solutions, Eq. (4.7) implies

$$\{h, v\} = K = \text{const.}$$

Using formulae (4.1) and Schwarz' inequality, we find

$$|K| = |h_x v_y - h_y v_x| \leq \frac{\sqrt{\mu\rho}}{2}\left(v_x^2 + v_y^2 + \frac{h_x^2 + h_y^2}{\mu\rho}\right) \leq \frac{\sqrt{\mu\rho}}{2}(\mathbf{V}^2 + B^2/\mu\rho). \tag{4.10}$$

We consider below only physically meaningful MHD equilibria for which the total kinetic and magnetic energy in every layer $z_1 < z < z_2$ is finite:

$$\frac{1}{2}\int_{z_1}^{z_2} \mathrm{d}z \int\int_{-\infty}^{\infty} (\rho\mathbf{V}^2 + \mathbf{B}^2/\mu)\mathrm{d}x\mathrm{d}y < \text{const.} \tag{4.11}$$

Hence we find that at least for some sequence of $x, y \longrightarrow \infty$ we have

$$\mathbf{V}^2 + \mathbf{B}^2/\mu\rho \longrightarrow 0.$$

Therefore the inequalities (4.10) yield that for the physically meaningful MHD equilibria $K = 0$. Hence the equation

$$\{h, v\} = 0 \tag{4.12}$$

holds. In view of (4.12), the stream function $v(x, y)$ and the flux function $h(x, y)$ are functionally dependent and hence $v = v(u)$, $h = h(u)$, where $u = u(x, y)$ is a smooth function.

Theorem 2 *The steady equations of magnetohydrodynamics have the following Lie point symmetry applicable to the translationally invariant equilibria satisfying physical condition (4.11). Let vector fields $\mathbf{V}(x, y)$ and $\mathbf{B}(x, y)$ (4.1) and pressure $p(x, y)$ satisfy the steady equations (2.1)–(2.2). Let $v_1(u)$ and $h_1(u)$ be arbitrary smooth functions satisfying (2.17). The symmetry transforms the solution $\mathbf{V}(x, y)$, $\mathbf{B}(x, y)$, $p(x, y)$ into the new solution*

$$\begin{aligned} &\mathbf{V}_1 = \tfrac{v_1'}{v'}\mathbf{V} + \left(f_1 - \tfrac{v_1'}{v'} f\right)\mathbf{e}_z, \quad \mathbf{B}_1 = \tfrac{h_1'}{h'}\mathbf{B} + \left(g_1 - \tfrac{h_1'}{h'} g\right)\mathbf{e}_z, \\ &p_1 = Cp + \tfrac{\rho}{2}(C\mathbf{V}^2 - \mathbf{V}_1^2) - \tfrac{\rho}{2}(Cf^2 - f_1^2) + \tfrac{1}{2\mu}(Cg^2 - g_1^2), \end{aligned} \tag{4.13}$$

where $f_1(u)$ and $g_1(u)$ are arbitrary functions of u.

Proof Indeed, Eqs. (4.5)–(4.8), (4.12) for the steady case take the form

$$\{h, v\} = 0, \quad \{\Phi, v\} - \frac{1}{\mu\rho}\{\Psi, h\} = 0, \tag{4.14}$$

$$\{f, v\} - \frac{1}{\mu\rho}\{g, h\} = 0, \quad \{g, v\} - \{f, h\} = 0. \tag{4.15}$$

The equation $\{h, v\} = h_x v_y - h_y v_x = 0$ means that the mapping $(x, y) \longrightarrow (h(x, y), v(x, y))$ has zero Jacobian. Hence its image is a curve Γ_1 in the plane (h, v). Let this curve satisfy an equation such as (2.16). We suppose that (2.16) is resolved in parametric form $h = h(u)$, $v = v(u)$ where $h(u)$ and $v(u)$ are functions of one variable u, and $u = u(x, y)$ is a function of two variables x and y.

The second equation of (4.14) has the form

$$\left(v'^2 - \frac{1}{\mu\rho}h'^2\right)\{u_{xx} + u_{yy}, u\} + \left(v'v'' - \frac{1}{\mu\rho}h'h''\right)\left\{(u_x)^2 + (u_y)^2, u\right\} = 0. \tag{4.16}$$

Let us consider another set of functions $v_1 = v_1(u)$, $h_1 = h_1(u)$ instead of the functions $v(u)$, $h(u)$. It is evident that the corresponding equation (4.16) is equivalent to the original equation (4.16) if the functions $v_1(u)$ and $h_1(u)$ satisfy (2.17).

After substituting $v = v(u)$, $h = h(u)$, Eq. (4.15) takes the form

$$\{v'(u)f - \frac{1}{\mu\rho}h'(u)g, u\} = 0, \quad \{h'(u)f - v'(u)g, u\} = 0.$$

These two equations imply $f = f(u)$, $g = g(u)$ with arbitrary functions $f(u)$ and $g(u)$, provided that $m(u) = (v'(u))^2 - (h'(u))^2/\mu\rho \neq 0$.

Hence we obtain that, if functions $v(u)$, $h(u)$, $f(u)$, $g(u)$ in a steady solution $\mathbf{V}(x, y)$, $\mathbf{B}(x, y)$ (4.1) are changed to functions $v_1(u)$, $h_1(u)$ and arbitrary functions $f_1(u)$, $g_1(u)$, then the corresponding vector fields $\mathbf{V}_1(x, y)$, $\mathbf{B}_1(x, y)$ (4.1) form a new solution to the magnetohydrodynamics equations (2.1), (2.2) provided that (2.17) holds. This transformation has the exact form (4.13) and constitutes the Lie point symmetry for the steady equations (2.1)–(2.2).

To derive formula (4.13) for the pressure $p_1(x, y)$, we consider the steady equations (4.3), (4.4). A direct calculation using the formulae (4.2) proves that, if functions $v(u)$ and $h(u)$ are changed to the functions $v_1(u)$, $h_1(u)$ satisfying (2.17), then $\mathcal{P}_1 - D_1/2 = C(\mathcal{P} - D/2)$. Hence formula (4.13) for the pressure $p_1(x, y)$ follows. □

Remark 7 Equations (4.1) and (4.12) imply

$$h'V - v'B = (h'f - v'g)\mathbf{e}_z.$$

Excluding the vector $\mathbf{e}_z$, we find that transform (4.13) also has the form

$$\mathbf{V}_1 = \frac{1}{k}(h'f_1 - v_1'g)\mathbf{V} + \frac{1}{k}(v_1'f - v'f_1)\mathbf{B}, \tag{4.17}$$
$$\mathbf{B}_1 = \frac{1}{k}(h'g_1 - h_1'g)\mathbf{V} + \frac{1}{k}(h_1'f - v'g_1)\mathbf{B},$$

where $k = h'f - v'g$. Transform (4.17) is evidently linear with respect to the vectors $\mathbf{V}$ and $\mathbf{B}$. Its coefficients depend on the function $u(x, y)$ and therefore are not constant on magnetic surfaces. Hence the transform (4.13)–(4.17) is different from the general symmetry transforms for the MHD equilibria [8,9] where coefficients are constant on magnetic surfaces. Since the magnetic surfaces are not translationally invariant in general, the symmetry transforms of [8,9] break the translational invariance of the MHD equilibria. Transforms (4.13)–(4.17) do preserve translational invariance. Note that the determinant of transform (4.17) is not constant since it is equal to

$$\frac{h_1'(u)f_1(u) - v_1'(u)g_1(u)}{h'(u)f(u) - v'(u)g(u)},$$

where $f_1(u)$, $g_1(u)$, $f(u)$ and $g(u)$ are arbitrary functions of function $u(x, y)$. For the symmetry transforms of [8,9], the corresponding determinant is constant, as well as the determinant (2.32) for the transforms (2.31) for the axially symmetric equilibria.

4.2 Reduction to the Klein–Gordon equation

The second equation of (4.14) has the form

$$\left\{v'(u)\Phi - \frac{1}{\mu\rho}h'(u)\Psi, u\right\} = 0.$$

As above, this equation implies

$$v'(u)\Phi - \frac{1}{\mu\rho}h'(u)\Psi = a_0(u), \tag{4.18}$$

where $a_0(u)$ is an arbitrary function. Substituting formulae (4.2), we derive the equation

$$\left(v'^2 - \frac{1}{\mu\rho}h'^2\right)(u_{xx} + u_{yy}) + \left(v'v'' - \frac{1}{\mu\rho}h'h''\right)\left((u_x)^2 + (u_y)^2\right) = a_0(u). \tag{4.19}$$

Let S_j be a segment of the curve Γ_1 (2.16) between two adjacent points u_j and u_{j+1} where $m(u_\ell) = 0$. We choose a new parametrization on S_j satisfying condition (3.6). Under this condition, Eq. (4.19) is reduced to the Klein–Gordon equation

$$u_{xx} + u_{yy} = a(u), \tag{4.20}$$

where $a(u) = a_0(u)/\beta$ is an arbitrary function. Equation (4.20) is well-known in the theory of equilibria of an ideal incompressible fluid. It describes also purely magnetic plasma equilibria with $\mathbf{V} = 0$.

Any solution to (4.20) defines a solution to the magnetohydrodynamics equations by the formulae

$$\begin{aligned} \mathbf{V} &= -v'(u)u_y\mathbf{e}_x + v'(u)u_x\mathbf{e}_y + f(u)\mathbf{e}_z, \\ \mathbf{B} &= -h'(u)u_y\mathbf{e}_x + h'(u)u_x\mathbf{e}_y + g(u)\mathbf{e}_z, \end{aligned} \tag{4.21}$$

where $f(u)$ and $g(u)$ are arbitrary functions and equation $(v'(u))^2 - (h'(u))^2/\mu\rho = \beta$ (3.6) holds.

Using formula (4.18), we transform the steady equations (4.3), (4.4) to the form

$$a_0(u)u_x = (\mathcal{P} - D/2)_x, \quad a_0(u)u_y = (\mathcal{P} - D/2)_y,$$

where $\mathcal{P} = p/\rho + \mathbf{V}^2/2$ and $D = f^2(u) - g^2(u)/\mu\rho$. Hence we obtain a formula for the pressure $p(x, y)$:

$$\frac{p}{\rho} = F(u) - \frac{g^2(u)}{2\mu\rho} - \frac{(v'(u))^2}{2}\left((u_x)^2 + (u_y)^2\right), \tag{4.22}$$

where $F'(u) = \beta a(u)$.

A direct substitution of formulae (4.21)–(4.22) in the magnetohydrodynamics equations (2.1)–(2.2) proves that these eight equations for the steady and z-independent case are reduced to the single equation (4.20) provided that condition (3.6) holds.

5 Conclusion

For axially symmetric MHD equilibria, we have derived the restricted Lie point symmetries (2.18), (2.27) which depend on arbitrary functions $v_1(u)$ and $h_1(u)$ satisfying (2.17) with an arbitrary parameter C. The symmetries are applicable to the physically meaningful MHD equilibria with bounded total energy in any layer $z_1 < z < z_2$; see (2.14) and (4.11). For translationally invariant MHD equilibria, the restricted Lie point symmetries have the form (4.13), (4.17). Applying the symmetries (2.18), (2.27) to the exact plasma equilibria presented in [14,15], one obtains large families of exact axially symmetric MHD equilibria. We have derived a reduction of eight MHD equilibrium equations to a single second-order partial differential equation (3.8) for the axially symmetric solutions and Eq. 4.20 for the translationally invariant ones.

For the future development in the field it would be useful to find the restricted Lie point symmetries for helically symmetric MHD equilibria and the corresponding reduction of the MHD equilibrium equations to a single partial differential equation of second order.

The restricted Lie point symmetries derived in this paper are different from the general symmetry transforms introduced in [7–9]. The latter are applicable to any MHD equilibria and have coefficients that are constant on magnetic surfaces but depend on a transversal variable and therefore can break the geometric symmetry (axial, translational or helical) of the equilibria. The transforms (2.18), (2.27) and (4.13), (4.17) preserve the geometric symmetry of the MHD equilibria.

Acknowledgements The author thanks the referee for useful remarks.

References

1. Kruskal MD, Kulsrud RM (1958) Equilibrium of a magnetically confined plasma in a toroid. Phys Fluids 1:265–274
2. Spitzer L (1958) The stellarator concept. Phys Fluids 1:253–264
3. Grad H, Rubin H (1958) Hydromagnetic equilibria and force-free fields, In: Proceedings of the second United Nations international conference on the peaceful uses of atomic energy, 31. United Nations, Geneva, pp 190–197
4. Shafranov VD (1958) On magnetohydrodynamical equilibrium configurations. Soviet Phys JETP 6:545–554
5. Chandrasekhar S (1956) Axisymmetric magnetic fields and fluid motions. Astrophys J 124:232–243
6. Chandrasekhar S, Prendergast KH (1956) The equilibrium of magnetic stars. Proc Natl Acad Sci USA 42:5–9
7. Bogoyavlenskij OI (2000) Helically symmetric astrophysical jets. Phys Rev E 62:8616–8627
8. Bogoyavlenskij OI (2002) Symmetry transforms for ideal magnetohydrodynamics equilibria. Phys Rev E 66(056410):1–11
9. Bogoyavlenskij OI (2002) Method of symmetry transforms for ideal MHD equilibrium equations. Contemp Math 301:195–218
10. Gardner CS, Greene JM, Kruskal MD, Miura RM (1967) Method for solving the Korteweg–de Vries equation. Phys Rev Lett 19:1095–1097
11. Hirota R (1971) Exact N-soliton of the Korteweg - de Vries equation for multiple collision of solitons. Phys Rev Lett 27:1192–1193
12. Ablowitz MJ, Kaup DJ, Newell AC, Segur H (1973) Method for solving the sine-Gordon equation. Phys Rev Lett 30:1262–1264
13. Alfven H, Falthammar C-G (1963) Cosmical electrodynamics. Fundamental principles. Clarendon Press, Oxford
14. Bogoyavlenskij OI (2000) Astrophysical jets as exact plasma equilibria. Phys Rev Lett 84:1914–1917
15. Bogoyavlenskij OI (2000) Counterexamples to Parker's theorem. J Math Phys 41:2043–2057